Radiative Transfer and Scattering

Previously we have only considered emission and absorption of microwaves. We must now include scattering:

\[ T_b(r, \hat{s}) \rightarrow T_b(r, \hat{s}) + dT_b(r, \hat{s}) \]

\[ dT_b(r, \hat{s}) = -k_e T_b(r, \hat{s}) \, dr + k_a T(r) \, dr + \int P(\hat{s}, \hat{s}') T_b(r, \hat{s}') \, d\hat{s}' \, dr \]

The change in brightness due to absorption and scattering:

- Extinction: \( k_e T_b(r, \hat{s}) \, dr \)
- Emission: \( k_a T(r) \, dr \)
- Scattering of brightness from another direction \( \hat{s}' \) into the beam \( \hat{s} \) of the original brightness \( T_b \).

Where

- \( k_e \): extinction coefficient
- \( k_e = k_a + k_s \)
- \( k_a \): absorption coefficient, represents the fraction of radiation that is absorbed in \( dr \)
- \( k_s \): scattering coefficient, represents the fraction of radiation that is scattered from the beam into a new direction

\[ P(\hat{s}, \hat{s}') = \text{phase function, describes how much brightness from the } \hat{s}' \text{ direction is scattered into } \hat{s} \]

\[ = \frac{k_s}{4\pi} 4(\hat{s}, \hat{s}') \]

\[ 4(\hat{s}, \hat{s}') = \text{normalized phase function} \]

Water vapor, oxygen, other gases \( \rightarrow k_a \)
Clouds and light precipitation \( \rightarrow k_a \) and \( k_s \)
Heavy precipitation (large hydrometeors) \( \rightarrow k_s \)

Assume:
- Local thermodynamic equilibrium, random positioning of hydrometeors (no coherent effects),
- No interaction among hydrometeors (distance between hydrometeors >> r).
Consider the following. The scattering particle is at the center of a spherical coordinate system and it is illuminated by a plane wave of power density $P_{i}$ along $\theta_{i}$.

If $P_{inc}$ = power associated with incident plane wave
$P_{inc}'$ = power density of incident plane wave

\[ P_{inc}' = \frac{|E_{i}(\theta_{i})|^{2}}{2\pi\varepsilon_{0}} \]

$E_{sca}(\theta_{i}) = \frac{-j\kappa R}{R} \frac{S_{np}(\theta_{i}, \theta_{i})}{R} E_{inc}(\theta_{i})$

a component of what is called the scattering matrix

\[ P_{sca} = \frac{\text{power density}}{2\pi\varepsilon_{0}} = \frac{|S_{np}(\theta_{i}, \theta_{i})|^{2}}{R^{2}} P_{inc}' \]

where $\int_{0}^{2\pi} |S_{np}(\theta_{i}, \theta_{i})|^{2} = m^{2}$

Scattering cross section

The total power scattered by a particle over a spherical surface of radius $R$ is:

\[ P_{sca} = R^{2} \int_{4\pi} P_{sca}(\theta_{i}) d\Omega = \int_{4\pi} |S_{np}(\theta_{i}, \theta_{i})|^{2} P_{inc}' dS_{\Omega} = P_{inc}' \int_{4\pi} |S_{np}(\theta_{i}, \theta_{i})|^{2} dS_{\Omega} \]

think of this quantity as brightness integrate over all scattered directions

\[ \sigma_{snp} = \text{scattering cross section for } p\text{-polarized incident radiation} \]
\[ = \frac{\text{power scattered}}{\text{incident power density}} = \frac{P_{sca}}{P_{inc}'} \]

\[ = \int_{4\pi} |S_{np}(\theta_{i}, \theta_{i})|^{2} d\Omega \]

\[ [\sigma_{snp}] = m^{2} \]

Examples
\[ \sigma_{snp}^{lh} = \text{scattering cross section for } l\text{-polarized incident radiation and } h\text{-polarized scattered radiation} \]
\[ \sigma_{snp}^{vh} = \text{scattering cross section for } h\text{-pol incident, } v\text{-pol scattered radiation} \]
Now think about this in terms of brightness...

\[
\int \frac{W}{sr} d\Omega' = W
\]

think of the particle occupying only a point in space, but also having an effective "area"

\[
\sigma_{\text{scat}}(\theta_i) = \frac{\text{q-pole scattered power}}{\text{p-pole incident power density}} \text{ over all solid angles}
\]

\[
\sigma_{\text{scat}}(\theta_i) = \frac{\text{p-pole reflected power density}}{\text{incident solid angle}}
\]

\[
\sigma_{\text{scat}}(\theta_i) = \sigma_{\text{scat}}(\theta_i) + \sigma_{\text{scat}}(\theta_i) + \sigma_{\text{scat}}(\theta_i) + \sigma_{\text{scat}}(\theta_i)
\]

the total unpolarized scattered power

the total unpolarized incident power density

\[
\sigma_{\text{abs}}(\theta_i) = \sigma_{\text{abs}}(\theta_i)
\]

Similarly,

the total unpolarized power absorbed by a scattering particle

the total unpolarized incident power density

\[
\sigma_{\text{abs}}(\theta_i) = \sigma_{\text{abs}}(\theta_i)
\]

\[
\sigma_{\text{abs}}(\theta_i) = \sigma_{\text{abs}}(\theta_i)
\]

Note that \( \sigma_{\text{scat}}(\theta_i) = Q_s \) in UMF!
Multiple Particle Scattering

Typically there is not just one particle scattering and absorbing radiation.

\[ N = \text{number of identical particles per volume} \]

\[ K_s(\Omega) = \text{scattering coefficient} = N_\text{sat}(\Omega) \]

\[ [K_s] = \frac{\text{m}^2}{\text{m}^3} = N_p \cdot \text{m}^{-1} \]

\[ K_a(\Omega) = \text{absorption coefficient for particles} = N_\text{abs}(\Omega) \]

\[ P(\Omega, \Omega') = \text{phase function} = N \left\{ |S_{gg}(\Omega, \Omega')|^2 + |S_{ga}(\Omega, \Omega')|^2 + |S_{ag}(\Omega, \Omega')|^2 + |S_{aa}(\Omega, \Omega')|^2 \right\} \]

\[ = \frac{K_s}{4\pi} 4(\Omega, \Omega') \quad 4(\Omega, \Omega') = \text{normalized phase function} \]

Summary for scattering and radiative transfer

\[ T_B(r, \Omega) = T_B(0, \Omega) e^{-\tau(0, r)} + \int_0^r \left\{ K_e(r') T(r') + \frac{K_s}{4\pi} \int 4(\Omega, \Omega') T_B(r', \Omega') d\Omega' \right\} e^{-\tau(r', r)} dr' \]

where:

\[ T(r_1, r_2) = \int_{r_1}^{r_2} K_e(r) dr \quad K_a = N_\text{abs} \quad K_s = N_\text{sat} \]

\[ \omega = \frac{K_s}{K_e} = \text{"scattering" albedo} \]

\[ K_e = K_a + K_s \]

\[ K_a = K_{ab} + K_{sp} \]

\[ K_{ab} = 2K_o n'' \quad \text{where} \quad n'' = \text{Im}(n^2) \quad n: \text{index of refraction of background medium} \]

\[ K_{sp} = \text{expressions for water vapor, oxygen, etc., absorption when background medium is Earth's atmosphere.} \]
Radar Backscattering Cross Section

The bistatic scattering cross section is defined for both a specific incidence direction and scatter direction, and defined as if the particle scatters the same amount of power density in all directions.

\[
P_{\text{Pinc}}(\hat{s}_i) = P_{\text{Pinc}}(\hat{s}_i) \int |S_{qp}(\hat{s}_s, \hat{s}_i)|^2 d\Omega_s = P_{\text{Pinc}} \left| S_{qp}(\hat{s}_s, \hat{s}_i) \right|^2 \int d\Omega_s
\]

\[
= 4\pi P_{\text{Pinc}} \left| S_{qp}(\hat{s}_s, \hat{s}_i) \right|^2
\]

\[
\sigma_{\text{bistatic}}(\hat{s}_s, \hat{s}_i) = 4\pi \left| S_{qp}(\hat{s}_s, \hat{s}_i) \right|^2 = \frac{P_{\text{Vscat}}(\hat{s}_s)}{P_{\text{Pinc}}(\hat{s}_i)} [\text{in m}^2]
\]

\[
\sigma_{\text{b}} = \frac{\text{radar backscattering cross section}}{\text{used in radar meteorology}} = \text{hypothesised total power scattered if scattering were isotropic and equal to the level of scattering in the backscattering direction, } \hat{s}_s = -\hat{s}_i
\]

\[
\sigma_{\text{b}} = 4\pi \left| S_{qp}(\hat{s}_s = -\hat{s}_i, \hat{s}_i) \right|^2 + 4\pi \left| S_{pp}(\hat{s}_i, \hat{s}_i) \right|^2
\]

\[
+ 4\pi \left| S_{pq}(\hat{s}_i, \hat{s}_i) \right|^2 + 4\pi \left| S_{qp}(\hat{s}_i, \hat{s}_i) \right|^2
\]

\[
= 4\pi R^2 \frac{P_{\text{Pinc}}(\hat{s}_s = -\hat{s}_i)}{P_{\text{Pinc}}(\hat{s}_i)}
\]
Example of single particle scattering: Scattering from a sphere

Mie theory gives exact, analytical expressions for scattering by a sphere.

Consider a metal sphere of radius \( r \) in a background of air.

\[
\xi_b = \text{backscattering efficiency} = \frac{\text{normalized radar backscattering cross section}}{\text{sphere cross section}}
\]

\[
\therefore \xi_b = \frac{\sigma_b}{\pi r^2}
\]

How does the effective scattering area compare to its physical or "real" area?

See Figure 5.15 from UMF.

\( \xi_b \) is given as a function of \( \chi = \frac{k r}{\lambda_0} = \frac{2\pi r}{c} \).

Note the transition from the Rayleigh region \( \chi = k r < 0.5 \) where the wavelength of observation is much larger than the sphere \( (2\pi r < 0.5) \), to the Mie region \( (\chi > 4\pi r) \), and finally to the optical region where \( r \gg \lambda_0 \) and \( \xi_b \rightarrow 1 \), or in other words, \( \sigma_b = \pi r^2 = \) physical or "real" area!

When the scattering particle is much smaller than the wavelength, a Rayleigh scattering approximation \( \xi \) the exact Mie theory can be used. Fortunately, this is often the case for hydrometeors in microwave remote sensing \( (i.e., r << \lambda_0) \).

Rayleigh scattering approximation (see Section 5-8.2 in UMF)

\[
\sigma_{\text{scat}}(\theta) = \frac{2k^6}{3\pi} \chi^6 |K|^2
\]

\[
\sigma_{\text{abs}}(\theta) = \frac{\lambda^2}{4} \chi^2 \text{Im}\{K\} - K^2
\]

\[
\Lambda = \frac{\lambda_0}{\text{Re}\{n_b^3\}} \text{ not clear, but this is what I think it should be}
\]

\( \sigma_{\text{abs}}(\theta) \) is normally larger than \( \sigma_{\text{scat}}(\theta) \) for the case of Rayleigh scattering.
Fig. 5.17  Mie efficiency factors for scattering and extinction by a water sphere as a function of drop radius, at 3 GHz (Fraser et al., © 1975 Am. Soc. Photogram.). Horizontal arrows indicate range of drop radii.

\( \lambda_0 = 10 \, \text{cm} \) (3 GHz)
\( n_w = 8.87 - j \, 0.626 \)
\( T = 273 \, \text{K} \)

Fig. 5.18  Mie efficiency factors for scattering and extinction by a water sphere as a function of drop radius, at 30 GHz (Fraser et al., © 1975 Am. Soc. Photogram.). Horizontal arrows indicate drop radii.

\( \lambda_0 = 1.0 \, \text{cm} \) (30 GHz)
\( n_w = 4.22 - j \, 2.52 \)
\( T = 273 \, \text{K} \)
Fig. 5.15  Radar backscattering efficiency as a function of $\chi$ for a metal sphere of radius $r$ (Skolnik, 1980).

Fig. 5.19  Mie efficiency factors for scattering and extinction by a water sphere as a function of drop radius, at 300 GHz (Fraser et al., © 1975 Am. Soc. Photogram.). Horizontal arrows indicate range of drop radii.