Uniform Plane Waves

A uniform plane wave is a solution to the Helmholtz equations in Cartesian coordinates. We can use a uniform plane wave to approximate the real spherical wave radiated by accelerating charges at long distances (the "far-field") away from the charges. We use time-harmonic (complex) fields to represent a plane wave for mathematical convenience.

For a complex field \( \mathbf{E} = E_0 e^{-jkz} \hat{x} \),
\[
E(x, z) = \text{the real field} = Re\{E_0 e^{-jkz} e^{j\omega t} \hat{x}\} = \times Re\{|E_0| e^{j\theta} e^{-jkz} e^{j\omega t} \hat{x}\} = \times |E_0| Re\{e^{j(kz-\omega t)}\} = |E_0| e^{j\omega (at-kz+\delta)}
\]

In general, a uniform plane wave has 2 orthogonal components which are both orthogonal to the direction of propagation.

If \( \hat{k} \) = direction of propagation = \( \hat{z} \), then component wave with amplitude in \( \hat{y} \) propagating in \( \hat{y} \)
\[
\mathbf{E} = \times \mathbf{E}_x + \hat{y} \mathbf{E}_y = \times E_{0x} e^{-jkz} + \hat{y} E_{0y} e^{-jkz}
\]

and \( \mathbf{H} = \frac{1}{j\omega} \times \mathbf{E} = \frac{1}{j\omega} \{ \hat{x} (\hat{y} \times \mathbf{E}_x) e^{-jkz} + (\hat{z} \times \mathbf{E}_y) e^{-jkz} \}
\]
\[
= \hat{x} \frac{\hat{y}}{\omega} E_{0y} e^{-jkz} - \hat{y} \frac{\hat{x}}{\omega} E_{0x} e^{-jkz} = \times \mathbf{H}_{0x} e^{-jkz} + \hat{y} \frac{\hat{z}}{\omega} E_{0z} e^{-jkz}
\]
\( \mathbf{H}_x \)
\( \mathbf{H}_y \)

You can think of a uniform plane wave as the sum of 2 waves, or as a single wave formed by the resultant vector. See Fig 7-6. Note that \( \mathbf{H} \) is always perpendicular to \( \mathbf{E} \).
\[
(\mathbf{H} = \frac{1}{j\omega} \hat{k} \times \mathbf{E})
\]
Wave Polarization

The polarization of a uniform plane wave describes the shape of the locus (collection of points) of the tip of the E vector at a given location in space as a function of time.

In this class we will only consider linear polarization. See Fig 7-12 for linear and other types of polarization.

Anthropogenic signals transmitted from radios, radars, etc. are polarized, i.e., they have a distinct polarization that is determined by the characteristics of the transmitting antenna.

Electromagnetic radiation naturally emitted by the atmosphere or land surface (or any material) is initially unpolarized which means the locus is random. Unpolarized radiation can still be decomposed into a pair of polarized components that are equal in magnitude.

- Antenna
  - Radar
  - Polarized radiation

- Antenna
  - Radiometer
  - Unpolarized radiation
  - Antenna is always favorable to a single type of polarization, acts like a filter, receiving one polarization and rejecting the other polarization.

  which normally "vertical polarization" or "horizontal polarization"

  this is a polarization pair
Plane Waves in Lossy Media

What happens when $k$ is complex, i.e., $\text{Im}\{k^2\} \neq 0$?

$$E = \hat{x} E_0 e^{-jkz} = \hat{x} E_0 e^{-j(k' - jk'')z}$$

where $k' = \text{Re}\{k^2\}$, $k'' = \text{Im}\{k^2\}$

$$E(t, z) = \text{Re}\{\hat{x} E_0 e^{j(k'-k'')z} e^{j\omega t}\}$$

$$= \text{Re}\{\hat{x} E_0 e^{-jkz} e^{j(\omega t - k'z + \theta)}\}$$

$$E(t, z) = \hat{x} |E_0| e^{-jkz} \cos(\omega t - k'z + \theta)$$

the imaginary part of $k$

causes the wave to be attenuated, i.e., the amplitude decreases in space!

When a medium through which a wave propagates is lossy, then the wave is attenuated as it propagates. A medium is lossy when $k = k_0 \sqrt{-\mu/\epsilon}$

is complex ($\text{Im}\{k^2\} \neq 0$). Since we assume in microwave remote sensing that $\mu_r = 1$, then a lossy medium has

$$\epsilon_r = \epsilon_r' - j \epsilon_r''$$

In general, when $\mu_r = 1$:

$$k' = k_0 |\text{Re}\{\sqrt{\epsilon_r}\}|, \quad k'' = k_0 |\text{Im}\{\sqrt{\epsilon_r}\}|$$

Define: $n = \text{index of refraction} = \sqrt{\epsilon_r} = n' - j n''$

Then,

$$n' = |n|^2 = \sqrt{\epsilon_r''} = \sqrt{\left(\sqrt{\epsilon_r'}^2 + (\epsilon_r''')^2 + \epsilon_r''\right) / 2}$$

$$n'' = \sqrt{(\epsilon_r'')^2 - \epsilon_r'}$$

Note that $k' = k_0 |\text{Re}\{n\}| = k_0 n'$, $k'' = k_0 |\text{Im}\{n\}| = k_0 n''$

Since $k' = f(n') = f(\epsilon_r', \epsilon_r'')$ and $k'' = f(n'') = f(\epsilon_r'')$

it is much simpler and more physical to use $n = \text{index of refraction}$, but unfortunately $\epsilon_r'$ is the standard way to characterize materials.

$k'' = f(n'')$ tells you how the wave is attenuated.

$k' = f(n')$ tells you about the wavelength in the material.
In a lossy medium, the magnitude of the electric field decreases exponentially with $z$ ($\equiv k=\alpha$).

\[
\frac{|E_x|}{|E_{0x}|} = e^{-kz} = e^{-k(\frac{1}{k_0})} = e^{-1} \approx 0.37
\]

$\delta_s \approx \frac{1}{k_0}$ = the "skin depth" of a medium, which is a measure of how far a wave propagates into the medium.

= the distance a wave travels before its magnitude is 37% ($\frac{1}{e}$) of its original magnitude.

**Electromagnetic Power Density**

The "Poynting vector" $\mathbf{S}$ is defined

\[
\mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad [\mathbf{S}] = \text{W}\cdot\text{m}^{-2}
\]

The direction of $\mathbf{S}$ is $k$!

$\mathbf{S}$ represents the instantaneous power per unit area carried by an electromagnetic wave.

The average power density is the time-average value of $\mathbf{S}$.

\[
\mathbf{S}_{\text{av}} = \text{average power density} = \frac{1}{2} \text{Re} \left\{ \mathbf{E} \times \mathbf{H}^* \right\}
\]

\[
[\mathbf{S}_{\text{av}}] = \text{W}\cdot\text{m}^{-2}, \quad ^* = \text{complex conjugate}
\]

A microwave radiometer measures $\mathbf{S}_{\text{av}}$ !!!

(Or at least the average power density & the polarization captured by the antenna of the radiometer.)
For a lossless medium, 
\[ E = (xE_0 + yE_0) e^{-\beta z}, \quad H = \frac{1}{i\omega} \times E = \frac{1}{i\omega} (xE_0 + yE_0) e^{-\beta z} \]

\[ E \times H^* = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ Ex_0 e^{-\beta z} & Ey_0 e^{-\beta z} & 0 \\ -\bar{E}^* e^{\beta z} & \bar{E}^* e^{\beta z} & 0 \end{vmatrix} = \delta(0) - \gamma(0) + 2 \left\{ \frac{Ex_0 E_0^*}{\omega} - E_0 \frac{\partial}{\partial \gamma} \left( 0 \right) \right\} = 2 \left\{ \frac{1}{2} \left[ |Ex_0|^2 + |Ey_0|^2 \right] \right\} \]

since \( AA^* = |A|^2 \)

So...

\[ S_{av} = \frac{1}{2} \text{Re} \left\{ E \times H^* \right\} \]

\[ = \frac{1}{2} \left\{ |E|^2 \right\} \quad \text{for a lossless medium} \quad (\text{Im} \{u\} = \text{Im} \{q\} = 0) \]

For a lossy medium, \( \gamma \) is also complex!

\[ E = (xE_0 + yE_0) e^{-\beta z} e^{-\gamma z}, \quad H = \frac{1}{i\omega} (-xE_0 + yE_0) e^{-\beta z} e^{-\gamma z} \]

\[ S_{av} = \frac{1}{2} \text{Re} \left\{ E \times H^* \right\} = \frac{1}{2} \left\{ \left| E(\gamma = 0) \right|^2 e^{-2\beta z} \right\} \text{Re} \left\{ \frac{\partial}{\partial \gamma} \right\} \]

\[ \text{Power} = \text{Re} \left\{ \frac{1}{\gamma} \left| \text{e}^{-\gamma z} \right|^2 \right\} = \frac{1}{m} \text{Re} \left\{ \text{e}^{-\beta z} \right\} = \frac{\cos \theta \alpha}{|m|} \]

\[ S_{av} = \frac{1}{2} \left\{ \left| E(\gamma = 0) \right|^2 e^{-2\beta z} \right\} \]

\[ \alpha = \text{power attenuation constant} \approx 2k = 2k_0 \nu \]

\[ \text{Power is attenuated at a rate of} \quad e^{-2k z} \]

This is what a radiometer measures!

\[ \text{The power associated with electromagnetic radiation is attenuated at a rate of} \quad e^{-2k z} \]
\[ E_x = E_{x_0} e^{-\frac{i}{\hbar} k z}, \quad E_y = E_{y_0} e^{-\frac{i}{\hbar} k z} \]
\[ E_{x_0} = a_x, \quad E_{y_0} = a_y e^{i \delta} \]

\( a_x, a_y, \delta \) produce different polarizations!

**Figure 7-6:** The wave \((E, H)\) is equivalent to the sum of two waves, one with fields \((E_x^+, H_y^+)\) and another with \((E_y^+, H_x^+)\), with both traveling in the +z-direction.

**Figure 7-7:** Linearly polarized wave traveling in the +z-direction (out of the page).

<table>
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<th>( \chi )</th>
<th>( \gamma )</th>
<th>-90°</th>
<th>-45°</th>
<th>0°</th>
<th>45°</th>
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<td>Right elliptical polarization</td>
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<td>-45°</td>
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<td>[Diagram]</td>
</tr>
</tbody>
</table>

**Figure 7-12:** Polarization states for various combinations of the polarization angles \((\gamma, \chi)\) for a wave traveling out of the page.
Figure 7-7: Linearly polarized wave traveling in the $+z$-direction (out of the page).

Figure 8-14: The plane of incidence is the plane containing the direction of wave travel, $\hat{k}$, and the surface normal to the boundary, which in the present case is the plane of the paper. A wave is (a) perpendicularly polarized when its $\mathbf{E}$ is perpendicular to the plane of incidence and (b) parallel polarized when its $\mathbf{E}$ lies in the plane of incidence.
Figure 7-11: Polarization ellipse in the x-y plane, with the wave traveling in the z-direction (out of the page).

Figure 7-9: Right-hand circularly polarized wave radiated by a helical antenna.

Figure 2.18 Rotation of a plane electromagnetic wave and its polarization ellipse at z = 0 as a function of time.
Figure 7-8: Circularly polarized plane waves propagating in the +z-direction (out of the page).

(a) LHC polarization

(b) RHC polarization

Figure 7-9: Right-hand circularly polarized wave radiated by a helical antenna.